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# Note

# Crisscross Latin Squares

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Let $A$ be a Latin square of order $n$. Then the $j$th right diagonal of $A$ is the set of $n$ cells of $A$:

$$\{(i, j+i): i = 0, 1, \ldots, n-1 (\bmod n)\};$$

and the $j$th left diagonal of $A$ is the set

$$\{(i, j-i): i = 0, 1, \ldots, n-1 (\bmod n)\}.$$

A diagonal is said to be complete if every element appears in it exactly once. For $n = 2m$ even, we introduce the concept of a crisscross Latin square which is something in between a diagonal Latin square and a Knut Vik design. A crisscross Latin square is a Latin square such that all the $j$th right diagonals for even $j$ and all the $j$th left diagonals for odd $j$ are complete. We show that a necessary and sufficient condition for the existence of a crisscross Latin square of order $2m$ is that $m$ is even.

## 1. Introduction

A *Latin square* of order $n$ is an $n \times n$ matrix such that each element in a set of $n$ elements appears exactly once in each row and each column. We index the rows, the columns, and the $n$ elements by the set $N = \{0, 1, \ldots, n-1\}$. Let $A$ denote a Latin square. Then the $j$th *right diagonal* of $A$ is the set of $n$ cells of $A$:

$$\{(i, j+i): i = 0, 1, \ldots, n-1\}$$

and the $j$th *left diagonal* is the set

$$\{(i, j-i): i = 0, 1, \ldots, n-1\},$$

where additions and subtractions are all mod $n$ in this paper unless otherwise specified. A diagonal is said to be *complete* if it consists of the set $N$.



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A Latin square $A$ is called a *diagonal Latin square* if both its 0th right and $(n-1)$st left diagonals are complete. $A$ is called a *Knut Vik design* if every right and left diagonal of $A$ is complete. It is known [1, 2, 5] that a necessary and sufficient condition for the existence of a diagonal Latin square of order $n$ is $n > 3$. It is also known [3, 4] that a necessary and sufficient condition for the existence of Knut Vik designs of order $n$ is that $n$ is not divisible by 2 or 3.

By the set of even (or odd) right (or left) diagonals, we mean all the $j$th right (or left) diagonals with $j$ even (or odd) where zero is always considered to be even. Let $A$ be a Latin square of order $n$ where $n = 2m$ is even. Then the union of the set of even right diagonals and the set of odd left diagonals consists of every cell of $A$ exactly once. We call $A$ a *crisscross Latin square* if all its even right diagonals and odd left diagonals are complete. Therefore, a crisscross Latin square is something in between a diagonal Latin square and a Knut Vik design, but is defined only for even order. In this paper, we show that a necessary and sufficient condition for the existence of a crisscross Latin square of order $2m$ is that $m$ is even. We give a simple and direct construction whenever a crisscross Latin square of order $n$ exists.

## 2. The Main Result

We first give a theorem which transforms the original problem into a more manageable form.

THEOREM 1. *A crisscross Latin square of order $n = 2m$ exists if and only if there exists a pair $(X, Y)$, $X = (x_0, x_1, \ldots, x_{2m-1})$, $Y = (y_0, y_1, \ldots, y_{2m-1})$ satisfying the following conditions*:

(i) *$X$ and $Y$ are permutations of the set* $(0, 1, \ldots, 2m-1)$.

(ii) $\{x_j - y_j : x_j \equiv y_j \pmod 2\} = \{0, 2, \ldots, 2m-2\}$.

(iii) $\{x_j + y_j : x_j \not\equiv y_j \pmod 2\} = \{1, 3, \ldots, 2m-1\}$.

*Proof*

*Necessity.* Let $A$ be a crisscross Latin square of order $2m$. Let $(x_j, y_j)$ be the cell containing the element 0 in the $j$th right (or left) diagonal if $j$ is even (or odd). Then from the definition of crisscross Latin square, the pair $(X, Y) = \{(x_j, y_j) : j = 0, 1, \ldots, 2m-1\}$ clearly satisfies the conditions of Theorem 1.

*Sufficiency.* Let $X^0 = (x_0, x_1, \ldots, x_{2m-1})$ and $Y^0 = (y_0, y_1, \ldots, y_{2m-1})$ be a pair satisfying the conditions of Theorem 1. Let $A$ be a square matrix of order $2m$. Suppose we assign the element 0 to the $n$ cells $(x_j, y_j)$, $j = 0, 1, \ldots, 2m-1$, of $A$. Then it is clear that each row, each column, each even right diagonal, and each odd left diagonal contains the element 0 exactly once. Now if we can find $2m - 1$ more pairs $(X^i, Y^i)$, $i = 1, 2, \ldots, 2m - 1$, all satisfying the conditions of Theorem 1 and furthermore, if the $n^2$ cells from the $2m$ pairs $(X^i, Y^i)$ are all distinct, then by assigning the element $i$ to the cells in $(X^i, Y^i)$, we obtain a crisscross Latin square.

We now give $2m$ such pairs. Let

$$(X^i, Y^i) = (X^0 + i, Y^0) \qquad \text{for} \quad i \text{ even},$$

and let

$$(X^i, Y^i) = (i - X^0, Y^0) \qquad \text{for} \quad i \text{ odd},$$

where $c \pm X^k$ means that $c$ is added to every number in $\pm X^k$. Then it is clear that each pair $(X^i, Y^i)$, $i = 0, 1, \ldots, 2m - 1$, satisfies the conditions of Theorem 1. Furthermore, in each column the cells in the pairs $(X^i, Y^i)$ for even $i$ occupy all the even (or odd) positions and the cells in the pairs $(X^i, Y^i)$ for odd $i$ occupy all the odd (or even) positions. Therefore the $n^2$ cells in the $2m$ pairs are distinct.

We are now ready to prove the main result of this paper.

THEOREM 2. *A necessary and sufficient condition for the existence of a crisscross Latin square of order $2m$ is that $m$ is even.*

*Proof*

*Necessity.* Let $(X, Y)$, $X = (x_0, x_1, \ldots, x_{2m-1})$, $Y = (y_0, y_1, \ldots, y_{2m-1})$, be a pair satisfying the conditions of Theorem 1. Without loss of generality, assume

$$x_i \equiv y_i (\text{mod } 2) \qquad \text{for} \quad i \text{ even},$$
$$x_i \not\equiv y_i (\text{mod } 2) \qquad \text{for} \quad i \text{ odd}.$$

Then

$$\begin{aligned}\sum_{j=0}^{2m-1} j &= \sum_{j=0}^{m-1} (x_{2j} - y_{2j}) + \sum_{j=0}^{m-1} (x_{2j+1} + y_{2j+1}) \\ &= \sum_{j=0}^{2m-1} x_j + \sum_{j=0}^{2m-1} y_j - 2 \sum_{j=0}^{m-1} y_{2j} \\ &= 2 \sum_{j=0}^{2m-1} j - 2 \sum_{j=0}^{m-1} y_{2j} ;\end{aligned}$$

or equivalently,

$$2 \sum_{j=0}^{m-1} y_{2j} = \sum_{j=0}^{2m-1} j = m(2m - 1).$$

Since $2\sum_{j=0}^{m-1} y_{2j}$ is even, $m$ must be even.

*Sufficiency.* For $m$ even, let $(X, Y) = \{(x_j, y_j): j = 0, 1,..., 2m-1\}$ consist of the following cells:

$$\begin{aligned}
(x_j, y_j) &= (2j, j) && \text{for } j \text{ even and } m-1 \geqslant j \geqslant 0,\\
(x_j, y_j) &= (2j+1, j+1) && \text{for } j \text{ even and } 2m-1 \geqslant j \geqslant m,\\
(x_j, y_j) &= (2j+1, 2m-1-j) && \text{for } j \text{ odd and } m-1 \geqslant j \geqslant 0,\\
(x_j, y_j) &= (2j, 2m-j) && \text{for } j \text{ odd and } 2m-1 \geqslant j \geqslant m.
\end{aligned}$$

Then it is easy to verify that the pair $(X, Y)$ satisfies the conditions of Theorem 1.

Theorem 2 now follows immediately from Theorem 1.

## 3. Examples

In this section we give crisscross Latin squares of the three smallest orders constructed by our method. Let $C_n$ denote a crisscross Latin square of order $n = 2m$.

Example 1. $m = 2$.

$$\binom{X}{Y} = \begin{pmatrix} 0 & 3 & 1 & 2 \\ 0 & 2 & 3 & 1 \end{pmatrix}, \qquad C_4 = \begin{pmatrix} 0 & 2 & 3 & 1 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix}.$$

Example 2. $m = 4$.

$$\binom{X}{Y} = \begin{pmatrix} 0 & 3 & 4 & 7 & 1 & 2 & 5 & 6 \\ 0 & 6 & 2 & 4 & 5 & 3 & 7 & 1 \end{pmatrix}, \qquad C_8 = \begin{bmatrix} 0 & 2 & 4 & 6 & 7 & 1 & 3 & 5 \\ 1 & 7 & 5 & 3 & 2 & 0 & 6 & 4 \\ 2 & 4 & 6 & 0 & 1 & 3 & 5 & 7 \\ 3 & 1 & 7 & 5 & 4 & 2 & 0 & 6 \\ 4 & 6 & 0 & 2 & 3 & 5 & 7 & 1 \\ 5 & 3 & 1 & 7 & 6 & 4 & 2 & 0 \\ 6 & 0 & 2 & 4 & 5 & 7 & 1 & 3 \\ 7 & 5 & 3 & 1 & 0 & 6 & 4 & 2 \end{bmatrix}.$$

Example 3. $m = 6$.

$$\binom{X}{Y} = \begin{pmatrix} 0 & 3 & 4 & 7 & 8 & 11 & 1 & 2 & 5 & 6 & 9 & 10 \\ 0 & 10 & 2 & 8 & 4 & 6 & 7 & 5 & 9 & 3 & 11 & 1 \end{pmatrix},$$

$$C_{12} = \begin{bmatrix} 0 & 2 & 8 & 6 & 4 & 10 & 11 & 1 & 7 & 5 & 3 & 9 \\ 1 & 11 & 5 & 7 & 9 & 3 & 2 & 0 & 6 & 8 & 10 & 4 \\ 2 & 4 & 10 & 8 & 6 & 0 & 1 & 3 & 9 & 7 & 5 & 11 \\ 3 & 1 & 7 & 9 & 11 & 5 & 4 & 2 & 8 & 10 & 0 & 6 \\ 4 & 6 & 0 & 10 & 8 & 2 & 3 & 5 & 11 & 9 & 7 & 1 \\ 5 & 3 & 9 & 11 & 1 & 7 & 6 & 4 & 10 & 0 & 2 & 8 \\ 6 & 8 & 2 & 0 & 10 & 4 & 5 & 7 & 1 & 11 & 9 & 3 \\ 7 & 5 & 11 & 1 & 3 & 9 & 8 & 6 & 0 & 2 & 4 & 10 \\ 8 & 10 & 4 & 2 & 0 & 6 & 7 & 9 & 3 & 1 & 11 & 5 \\ 9 & 7 & 1 & 3 & 5 & 11 & 10 & 8 & 2 & 4 & 6 & 0 \\ 10 & 0 & 6 & 4 & 2 & 8 & 9 & 11 & 5 & 3 & 1 & 7 \\ 11 & 9 & 3 & 5 & 7 & 1 & 0 & 10 & 4 & 6 & 8 & 2 \end{bmatrix}.$$

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